Bounded t-structures and finitistic dimension for triangulated categories

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Finitistic dimension

Definition 1.

Let R be a noetherian ring. Then, the *small finitistic dimension* is defined to be,

$$\mathbf{findim}(R) := \sup\{\operatorname{proj.dim.}(M) \mid M \in \operatorname{mod}(R) \text{ and } \operatorname{proj.dim.}(M) < \infty\}$$

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Main Definition.

Let T be a triangulated category and G any object in T. Then,

$$\mathbf{findim}(\mathsf{T}, G) := \inf \left\{ n \mid \left\{ \Sigma^i G : i \geqslant 1 \right\}^{\perp} \subseteq \Sigma^n \langle G \rangle^{[0,\infty)} \right\}$$

where

$\left\{ \Sigma^i G : i \geqslant 1 \right\}^{\perp} = G^{\perp_{<0}} := \left\{ F \in \mathsf{T} \mid \operatorname{Hom}_{\mathsf{T}}(\Sigma^i G, F) = 0 \text{ for all } i \geqslant 1 \right\}$

and $\langle G \rangle^{[0,\infty)}$ is the smallest strictly full subcategory of T containing G which is closed under desuspensions, direct summands, and extensions.

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For a unital ring R, $\mathbf{findim}(\mathbf{K^b}(\mathrm{proj}\text{-}R), R) = \mathbf{findim}(R)$.

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Henning Krause. *The finitistic dimension of a triangulated category*. arXiv e-prints. 2024. arXiv: 2307.12671v2

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Example 3.

Let Λ be an Artin algebra. Then,

$$\mathbf{findim}(\mathbf{D^b}(\text{mod-}\Lambda), S) < \infty \text{ and } \mathbf{findim}(\mathbf{D}_{\text{sg}}(\Lambda), S) < \infty$$

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Remark 4.

The two definitions differ for the case $\mathbf{D^b}(\text{mod-}\Lambda)$ for an Artin algebra Λ with infinite global dimension.

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Theorem 5.

If T *has a strong generator* G*, then,* $\mathbf{findim}(\mathsf{T}, G) < \infty$.

More examples

Theorem 6.

Let X be a qcqs scheme which is covered by affine schemes $\mathbf{Spec}(R_i)$ such that $\mathbf{findim}(R_i) < \infty$. Then, $\mathbf{findim}(\mathbf{D}^{\mathrm{perf}}(X), G) < \infty$ for any classical generator G of $\mathbf{D}^{\mathrm{perf}}(X)$. In particular, for a noetherian finite dimensional scheme X, $\mathbf{findim}(\mathbf{D}^{\mathrm{perf}}(X), G) < \infty$.

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Theorem 6.

Let X be a qcqs scheme which is covered by affine schemes $\mathbf{Spec}(R_i)$ *such that* $\mathbf{findim}(R_i) < \infty$*. Then,* $\mathbf{findim}(\mathbf{D}^{\text{perf}}_{Z}(X), G) < \infty$ *for any classical generator G of* $\mathbf{D}^{\text{perf}}_{Z}(X)$ *and any closed subset Z with quasicompact complement. In particular, for a noetherian finite dimensional scheme X,* $\mathbf{findim}(\mathbf{D}^{\text{perf}}_{Z}(X), G) < \infty$ *for any closed subset Z.*

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Theorem 7.

If R is a noetherian, non-positive DG-ring with finite finitistic dimension, then, $\mathbf{findim}(\mathbf{Thick}(R), R) < \infty$*.*

Isaac Bird, Liran Shaul, Prashanth Sridhar, and Jordan Williamson. *Finitistic dimensions over commutative DG-rings*. arXiv e-prints. 2022. arXiv: 2204.06865v2

Bounded t-structures

Definition 8.

Let T be a triangulated category. A t-structure $(\mathsf{T}^{\leqslant 0}, \mathsf{T}^{\geqslant 0})$ is *bounded above* if

$$\mathsf{T} = \bigcup_{i \in \mathbb{Z}} \Sigma^i \mathsf{T}^{\leqslant 0}$$

Definition 8.

Let T be a triangulated category. A t-structure $(\mathsf{T}^{\leqslant 0}, \mathsf{T}^{\geqslant 0})$ is *bounded* if

$$\bigcup_{i\in\mathbb{Z}} \Sigma^i \mathsf{T}^{\geqslant 0} = \mathsf{T} = \bigcup_{i\in\mathbb{Z}} \Sigma^i \mathsf{T}^{\leqslant 0}$$

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Remark 9.

Let T be a triangulated category with a classical generator G. If $H \in \mathsf{T}$ such that $\mathbf{findim}(\mathsf{T}, H) < \infty$, then $\mathbf{findim}(\mathsf{T}, G) < \infty$. In such a case we say $\mathbf{findim}(\mathsf{T}) < \infty$.

Lemma 10.

Let T *be a triangulated category with an object* G *such that* $\mathbf{findim}(\mathsf{T}, G) < \infty$. *If* T *has a bounded t-structure, then* G *is a classical generator for* T.

Main Theorem A

Meta Theorem.

Let T be a triangulated category with an object G such that $\textbf{findim}(\mathsf{T}^{\mathrm{op}}, G) < \infty$. If T has a bounded t-structure, then the "singularity category of T" vanishes, that is T is "regular".

Main Theorem A.

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Reminder on metrics and completions

Definition 11 (Neeman).

A *good metric* on a triangulated category T is a sequence $\{\mathcal{N}_i\}_{i \in \mathbb{N}}$ of full subcategories of T containing 0 such that for all $i \in \mathbb{N}$,

1. $\mathcal{N}_i \star \mathcal{N}_i \subseteq \mathcal{N}_i$
2. $\Sigma^{-1}\mathcal{N}_{i+1} \cup \mathcal{N}_{i+1} \cup \Sigma\mathcal{N}_{i+1} \subseteq \mathcal{N}_i$

Remark 12.

Let T be a triangulated category T with an object $G \in \mathsf{T}$. Then, we can define the good metric $\left\{\Sigma^n \langle G \rangle^{(-\infty,0]}\right\}_{n \geqslant 0}$ which we call the *G-good metric*. Here $\langle G \rangle^{(-\infty,0]}$ is the striclty full subcategory generated by G which is closed under suspensions, direct summands, and extensions.

Remark 13 (Neeman).

Given a triangulated category T with a good metric, we can define the completion, $\mathfrak{S}(\mathsf{T})$ inside $\mathrm{Hom}(\mathsf{T}^{\mathrm{op}}, \mathrm{Ab})$. It is a triangulated category.

Examples of $\mathfrak{S}(\mathsf{T})$

Remark 14.

1. We denote the completion of a triangulated category T with respect to the G-good metric by $\mathfrak{S}_G(\mathsf{T})$.
2. For a triangulated category with a classical generator G, we will always consider the completion with respect to the G-good metric, unless stated otherwise.

Example 15 (Neeman).

1. Let R be a noetherian ring. Then,

$$\mathfrak{S}(\mathbf{K^b}(\text{proj-}R)) = \mathbf{D^b}(\text{mod-}R)$$

2. Let X be a noetherian scheme then,

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Corollaries

Main Theorem A.

Let T be a triangulated category with an object G such that $\mathbf{findim}(\mathsf{T}^{\mathrm{op}}, G) < \infty$. If T has a bounded t-structure, then $\mathsf{T} = \mathfrak{G}_G(\mathsf{T})$.

Corollary 16.

For an Artin algebra Λ such that $\mathbf{findim}(\Lambda^{\mathrm{op}}) < \infty$, $\mathbf{K^b}(\mathrm{proj}\text{-}\Lambda)$ has a bounded t-structure if and only if Λ has finite global dimension.

Corollary 17.

Let R be a ring such that $\mathbf{findim}(R^{\mathrm{op}}) < \infty$. Then,

1. *If R is noetherian (or even coherent) then there exists a bounded t-structure on $\mathbf{K^b}(\mathrm{proj}\text{-}R)$ if and only if $\mathbf{D_{sg}}(R) = 0$.*
2. *If there exists a bounded t-structure on $\mathbf{K^b}(\mathrm{proj}\text{-}R)$ then $\mathbf{K^b}(\mathrm{proj}\text{-}R) = \mathbf{K^{-,b}}(\mathrm{proj}\text{-}R)$.*

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Corollary for DG-rings

Lemma 18.

Let R be a non-positive noetherian DG-ring. Then,

$$\mathfrak{S}(\mathbf{Thick}(R)) = \mathbf{D}^{\mathbf{b}}_{\mathbf{f}}(R)$$

where

$$\mathbf{D}^{\mathbf{b}}_{\mathbf{f}}(R) = \left\{ F \mid \mathrm{H}^{i}(F) = 0 \text{ for all } |i| >> 0 \text{ and } \mathrm{H}^{i}(F) \in \mathbf{mod}(\mathrm{H}^{0}(R)) \forall i \in \mathbb{Z} \right\}$$

Corollary 19.

Let R be a non-positive noetherian DG-ring with $\mathbf{findim}(R^{\mathrm{op}}) < \infty$. If $\mathbf{Thick}(R)$ has a bounded t-structure then $\mathbf{Thick}(R) = \mathbf{D}^{\mathbf{b}}_{\mathbf{f}}(R)$.

Result in Algebraic Geometry

Conjecture 20 (Antieau, Gepner, Heller).

Let X be a noetherian finite-dimensional scheme. Then, $\mathbf{D}^{\mathsf{perf}}(X)$ has a bounded t-structure if and only if X is regular, that is $\mathbf{D}_{\mathsf{sg}}(X) = 0$.

Benjamin Antieau, David Gepner, and Jeremiah Heller. "K-theoretic obstructions to bounded t-structures". In: *Invent. Math.* 216.1 (2019), pp. 241–300

Result in Algebraic Geometry

Theorem 20 (Neeman).

Let X be a noetherian finite-dimensional scheme with a closed subset Z. Then, $\mathbf{D}^{\mathrm{perf}}_{Z}(X)$ has a bounded t-structure if and only if $Z \subseteq \mathrm{reg}(X)$, that is $\mathbf{D}^{\mathrm{perf}}_{Z}(X) = \mathbf{D}^{\mathbf{b}}_{\mathbf{coh},Z}(X)$.

Amnon Neeman. *Bounded t-structures on the category of perfect complexes*. Acta Math (to appear). arxiv:2202.08861v3

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Harry Smith. "Bounded t-structures on the category of perfect complexes over a Noetherian ring of finite Krull dimension". In: *Adv. Math* 399 (2022), 21pp

A brief digression on non-existence of (tensor) t-structures

Theorem 21 (Smith).

There exist no non-trivial t-structures on $\mathbf{K^b}(\operatorname{proj}\text{-}R) = \mathbf{D}^{\mathrm{perf}}(\mathbf{Spec}(R))$ *for a singular noetherian finite dimensional ring* R*.*

Uses the classification of compactly generated t-strcutures on $D(R)$.

Theorem 22 (Biswas, Parker, MR).

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Leovigildo Alonso Tarrío, Ana Jeremías López, and Manuel Saorín. "Compactly generated t-structures on the derived category of a Noetherian ring". In: *Journal of Algebra* 324 (2010), pp. 313–346

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Definition 23.

Let T be a triangulated category with a good metric $\{\mathcal{N}_i\}_{i\in\mathbb{N}}$. A t-structure $(\mathsf{T}^{\leqslant 0}, \mathsf{T}^{\geqslant 0})$ is *extendable* if there exists $i > 0$ such that $\mathcal{N}_i \subseteq \mathsf{T}^{\leqslant 0}$. In particular, any bounded t-structure is extendable for the metric given by $\left\{\Sigma^i \langle G \rangle^{(-\infty,0]}\right\}_{i\in\mathbb{N}}$ for any object $G \in \mathsf{T}$.

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Definition 24.

Let T be a triangulated category with a metric $\{\mathcal{N}_i\}$. Then, for any full subcategory A of T, we define, $\mathfrak{S}(\mathsf{A}) := \mathfrak{L}(\mathsf{A}) \cap \mathfrak{C}(\mathsf{T})$, where $\mathfrak{L}(\mathsf{A})$ is obtained by taking the colimits of all Cauchy sequences in A under the Yoneda embeddding.

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Theorem 24.

Let $(\mathsf{T}^{\leqslant 0}, \mathsf{T}^{\geqslant 0})$ be an extendable t-structute on a triangulated category T with a metric $\{\mathcal{N}_i\}$. Then,

1. $\left(\mathfrak{S}(\mathsf{T}^{\leqslant 0}), \mathfrak{S}(\mathsf{T}^{\geqslant 0})\right)$ *is a t-structure on $\mathfrak{S}(\mathsf{T})$. Further, the hearts of the two t-structure are equivalent via the Yoneda embedding.*
2. *If $(\mathsf{T}^{\leqslant 0}, \mathsf{T}^{\geqslant 0})$ is bounded above, so is $\left(\mathfrak{S}(\mathsf{T}^{\leqslant 0}), \mathfrak{S}(\mathsf{T}^{\geqslant 0})\right)$.*
3. *Let* **findim**$(\mathsf{T}^{op}, G) < \infty$. *Then, if $(\mathsf{T}^{\leqslant 0}, \mathsf{T}^{\geqslant 0})$ is bounded, so is $\left(\mathfrak{S}_G(\mathsf{T}^{\leqslant 0}), \mathfrak{S}_G(\mathsf{T}^{\geqslant 0})\right)$.*

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3. **Let findim**$(\mathsf{T}^{\mathrm{op}}, G) < \infty$**. Then, if** $(\mathsf{T}^{\leqslant 0}, \mathsf{T}^{\geqslant 0})$ **is bounded, so is** $\left(\mathfrak{S}_G(\mathsf{T}^{\leqslant 0}), \mathfrak{S}_G(\mathsf{T}^{\geqslant 0})\right)$.

Definition 25.

Let T be a triangulated category. Two t-structures $(\mathsf{T}_1^{\leqslant 0}, \mathsf{T}_1^{\geqslant 0})$ and $(\mathsf{T}_2^{\leqslant 0}, \mathsf{T}_2^{\geqslant 0})$ are *equivalent* if $\mathsf{T}_2^{\leqslant -i} \subseteq \mathsf{T}_1^{\leqslant 0} \subseteq \mathsf{T}_2^{\leqslant i}$ for some $i \geqslant 0$.

Theorem 26 (Neeman).

Let X be a noetherian finite-dimensional scheme. Then, all bounded t-structures on $\mathbf{D}^{\mathbf{b}}_{\mathbf{coh},\mathsf{Z}}(X)$ are equivalent if any of the following holds,

1. $\mathsf{Z} \subseteq \mathrm{reg}(X)$
2. *X has a dualizing complex.*
3. *$\mathsf{Z} = X$, and X is separated and quasiexcellent.*

Equivalence of t-structures

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Main Theorem B

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Let T be a triangulated category with an object G such that $\mathbf{findim}(\mathsf{T}^{\mathrm{op}}, G) < \infty$. Then, all bounded t-structures on any category $\mathcal{X}$ such that $\mathsf{T} \subseteq \mathcal{X} \subseteq \mathfrak{S}_G(\mathsf{T})$ are equivalent.

Corollary 27.

Let X be a noetherian finite-dimensional scheme with a closed subset Z. Then, all bounded t-structures on $\mathbf{D}^{\mathbf{b}}_{\mathbf{coh},Z}(X)$ are equivalent.

Corollary 28.

Let R be noetherian (or coherent) ring such that $\mathbf{findim}(R^{\mathrm{op}}) < \infty$. Then all bounded t-structures on any trianfulated category $\mathcal{X}$ such that $\mathbf{K}^{\mathbf{b}}(\mathrm{proj}\text{-}R) \subseteq \mathcal{X} \subseteq \mathbf{D}^{\mathbf{b}}(\mathrm{mod}\text{-}R)$ are equivalent.

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Some questions

Example 29.

Let $R = k[X_1, X_2, \cdots]$ for a field k. Recall that $\mathbf{findim}(R) = \infty$. Further,

1. $\mathbf{K^b}(\text{proj-}R) = \mathbf{D^b}(\text{mod-}R)$
2. There are two non-equivalent bounded t-structures on $\mathbf{D^b}(\text{mod-}R)$.

Question 30.

Let R be a coherent ring. Is it true that if $\mathbf{K^b}(\text{proj-}R)$ has a bounded t-structure then $\mathbf{D}_{\text{sg}}(R) = 0$?

Question 31.

In what generality is it true that if a classicaly generated triangulated category T has a bounded t-structure, then $\mathsf{T} = \mathfrak{S}(\mathsf{T})$?

Some questions

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